

Detecting the concurrence of an unknown state with a single observable

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While the detection of entanglement has been proved already to be quite a difficult task, experimental quantification of entanglement is even more challenging. In this work, we derive an analytical lower bound for the concurrence of a bipartite mix quantum state in arbitrary dimension. The lower bound is experimentally implementable in a feasible way, which enabling quantification of entanglement in a broad variety of cases.

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Entanglement is a distinctive feature of quantum mechanics [1, 2], and an indispensable ingredient in various kinds of quantum information processing applications vary from quantum cryptography [3] and quantum teleportation [4] to measurement-based quantum computing [5].

The use of entanglement as a resource not only bears the question of how it can be detected, but also how it can be quantified. For this purpose, several entanglement measures have been introduced, one of the most prominent of which is the concurrence [6].

However, calculation of the concurrence is a formidable task as the Hilbert space dimension is increasing. Good algorithms and progresses have been obtained concerning lower bounds [9–12]. Considerable progress is made in order to give a purely algebraic lower bound and experimental verifying it [13–23].

We start with a generalized definition of concurrence for a pure state $|\psi\rangle$ in the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of two Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ for systems A, B . The concurrence is defined by

$$C(|\psi\rangle) := \sqrt{2(1 - \text{Tr} \rho_A^2)} \quad (1)$$

where the reduced density matrix ρ_A is obtained by tracing over the subsystem B . The concurrence is then extended to mixed states ρ by the convex roof,

$$C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (2)$$

for all possible ensemble realizations $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. From definition, a state ρ is *separable* if and only if $C(\rho) = 0$.

In this letter, we will show that strong lower bounds of concurrence can be derived by exploiting close relations between concurrence and a recently introduced detection criteria for bipartite entanglement. The lower bound we found is not only analytical, but also is physical accessible, i.e., we can easily use the existing experiments technology to obtain this bound (e.g., [17–19, 25]).

To disclose the connection between concurrence and entanglement detection criteria, let us first review the

known criterions. Until now, many possible ways to detect entanglement have been proposed. These range from Bell inequalities, entanglement witnesses and spin squeezing inequalities to entropic inequalities, the measurement of nonlinear properties of the quantum state and the approximation of positive maps, see [1, 2].

Among so many methods, the PPT criterion and the realignment criterion are the two distinguished ones, they are most powerful and widely used by quantum information community. The Peres-Horodecki criterion of positivity under partial transpose (PPT criterion) [26, 27] say that $\rho^{T_A} \geq 0$ should be satisfied for a separable state, where ρ^{T_A} stands for a partial transpose with respect to the subsystem A . This criterion is even to be sufficient for 2×2 and 2×3 bipartite systems. However, for high dimension system, there exists entanglement states that are positive under partial transpose, the so called bound entanglement states (BES) [28].

Another complementary operational criterion for separability called the *realignment* criterion is very strong in detecting many of BES [29]. This criterion states that a realigned version $\mathcal{R}(\rho)$ of ρ should satisfy $\|\mathcal{R}(\rho)\| \leq 1$ for any separable state ρ . Note that realignment criterion is in some sense “dual” to the PPT criterion, in the following way: A density operator can be written as

$$\begin{aligned} \rho &= \sum_{ijkl} \langle ij|\rho|kl\rangle |ij\rangle\langle kl| \\ &= \sum_{ijkl} \rho_{ij,kl} |i\rangle\langle k| \otimes |j\rangle\langle l| \end{aligned} \quad (3)$$

A partial transpose with respect to the first system A is

$$(\rho^{T_A})_{ij,kl} := \rho_{kj,il}. \quad (4)$$

While the realignment of ρ is defined as:

$$(\rho^R)_{ij,kl} := \rho_{ik,jl}. \quad (5)$$

Recently, a criterion was introduced, which was strictly stronger than the realignment criterion. It says that, if a state is separable, then the following must hold [30]:

$$\|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\| \leq \sqrt{(1 - \text{Tr} \rho_A^2)(1 - \text{Tr} \rho_B^2)} \quad (6)$$

We now derive the main result of this Letter.

Theorem: For any $m \otimes n$ ($m \leq n$) mixed quantum state ρ , the concurrence $C(\rho)$ satisfies the following:

$$C(\rho) \geq \sqrt{\frac{2n}{(n-1)(n+1)^2}} \times f(\rho). \quad (7)$$

Where $f(\rho) := \|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\| - \sqrt{(1 - \text{Tr}\rho_A^2)(1 - \text{Tr}\rho_B^2)}$.

Proof.— Without loss of generality, we suppose that a pure $m \otimes n$ ($m \leq n$) quantum state $\rho := |\psi\rangle\langle\psi|$ has the standard Schmidt form

$$|\psi\rangle = \sum_i \sqrt{\mu_i} |a_i b_i\rangle, \quad (8)$$

where $\sqrt{\mu_i}$ ($i = 1, \dots, m$) are the Schmidt coefficients, $|a_i\rangle$ and $|b_i\rangle$ are orthonormal basis in \mathcal{H}_A and \mathcal{H}_B , respectively. The two reduced density matrices ρ_A and ρ_B have the same eigenvalues of μ_i . It follows

$$C^2(\rho) = 2\left(1 - \sum_i \mu_i^2\right) = 4 \sum_{i < j} \mu_i \mu_j, \quad (9)$$

which varies smoothly from 0, for pure product states, to $2(m-1)/m$ for maximally entangled pure states.

Then we get

$$\begin{aligned} & \|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\| \\ &= \left\| \sum_i (\mu_i - \mu_i^2) |a_i a_i\rangle \langle b_i b_i| + \sum_{i \neq j} \sqrt{\mu_i \mu_j} |a_i a_j\rangle \langle b_i b_j| \right. \\ & \quad \left. - \sum_{i \neq j} \mu_i \mu_j |a_i a_i\rangle \langle b_j b_j| \right\| \\ &\leq \left\| \sum_i (\mu_i - \mu_i^2) \right\| + \sum_{i \neq j} \sqrt{\mu_i \mu_j} + \sum_{i \neq j} \mu_i \mu_j \\ &= \sum_i (\mu_i - \mu_i^2) + \sum_{i \neq j} \sqrt{\mu_i \mu_j} + \sum_{i \neq j} \mu_i \mu_j \end{aligned}$$

So from Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\| - \sqrt{(1 - \text{Tr}\rho_A^2)(1 - \text{Tr}\rho_B^2)} \\ &\leq 2\sqrt{\mu_1 \mu_2} (1 + \sqrt{\mu_1 \mu_2}) + 2\sqrt{\mu_1 \mu_3} (1 + \sqrt{\mu_1 \mu_3}) \\ & \quad + \dots + 2\sqrt{\mu_{n-1} \mu_n} (1 + \sqrt{\mu_{n-1} \mu_n}) \\ &\leq 2 \sqrt{\sum_{i < j} \mu_i \mu_j} \sqrt{\sum_{i < j} (1 + \sqrt{\mu_i \mu_j})^2} \\ &\leq 2 \sqrt{\sum_{i < j} \mu_i \mu_j} \sqrt{\sum_{i < j} (1 + 2\sqrt{\mu_i \mu_j}) + \sum_{i < j} (\mu_i \mu_j)} \\ &\leq 2 \sqrt{\sum_{i < j} \mu_i \mu_j} \sqrt{\sum_{i < j} (1 + (\mu_i + \mu_j)) + \frac{n-1}{2n}} \\ &= 2 \sqrt{\sum_{i < j} \mu_i \mu_j} \sqrt{\frac{n(n-1)}{2} + (n-1) + \frac{n-1}{2n}} \\ &= \sqrt{\frac{(n-1)(n+1)^2}{2n}} \times C(\rho) \end{aligned}$$

Now, we will show that the inequality (7) also holds for mix state. Assume we have found the optimal decomposition $\sum_i p_i \rho_i$ for ρ to achieve the infimum of $C(\rho)$, where ρ_i are pure state density matrices. Then $C(\rho) = \sum_i p_i C(\rho_i)$ by definition. Now, we need to prove

that after mixture, the bound will not become bigger, i.e., $f(\sum_i p_i \rho_i) \leq \sum_i p_i f(\rho_i)$. From the convex property of the trace norm, we know that $\|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\|$ is decreasing after convex combination, so the only thing left to prove is that $\sqrt{(1 - \text{Tr}\rho_A^2)(1 - \text{Tr}\rho_B^2)}$ is increasing after convex combination. It is sufficient to consider the case of $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$. Now, we will prove that

$$\sqrt{[1 - \frac{1}{4}\text{Tr}(\rho_{1A} + \rho_{2A})^2][1 - \frac{1}{4}\text{Tr}(\rho_{1B} + \rho_{2B})^2]} \geq \frac{1}{2}[1 - \text{Tr}(\rho_{1A}^2)] + \frac{1}{2}[1 - \text{Tr}(\rho_{2A}^2)] \quad (10)$$

For clear, denote $\text{Tr}(\rho_{1A}^2) = \text{Tr}(\rho_{1B}^2) := x_1$, $\text{Tr}(\rho_{2A}^2) = \text{Tr}(\rho_{2B}^2) := x_2$, $\text{Tr}(\rho_{1A}\rho_{2A}) := x_3$, $\text{Tr}(\rho_{1B}\rho_{2B}) := x_4$, then the inequality (10) reduced to prove that the function $F := -\frac{3}{16}(x_1+x_2)^2 + \frac{1}{8}(x_1+x_2)(x_3+x_4) + \frac{1}{4}x_3x_4 - \frac{1}{2}(x_3+x_4) + \frac{1}{2}(x_1+x_2) \geq 0$. To get the minimal value of F , we will use the Lagrange multipliers method, and find that F attained its minimal value at the point $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$. And in this case, the minimal value is $F = 0$. So we get $F \geq 0$. Theorem is proved. ■

The most prominent feature of this theorem is that it not only allows to obtain a strong lower bound for the concurrence without any numerical optimization procedure, but also this bound is measurable, i.e., it is directly accessible in currently existing laboratory experiments.

We will explain this in detail. We will see that, our bound is experimentally implementally by means of local observables, use the method as that of [13, 18, 24, 25].

First, note that the concurrence of a bipartite pure state has another representation as [13, 18]

$$C(|\psi\rangle) \equiv \sqrt{2(1 - \text{Tr}\rho_A^2)} = \sqrt{\langle\psi| \otimes \langle\psi| A |\psi\rangle \otimes |\psi\rangle}, \quad (11)$$

where $A = 4P_-^{(1)} \otimes P_-^{(2)}$. $P_-^{(i)}$ is the projector on the antisymmetric subspace $\mathcal{H}_i \wedge \mathcal{H}_i$ of the two copies of the i th subsystem $\mathcal{H}_i \otimes \mathcal{H}_i$. Define $K_1 = 4P_-^{(1)} \otimes \mathbb{1}^{(2)}$ and $K_2 = 4(\mathbb{1}^{(1)} \otimes P_-^{(2)})$, then $1 - \text{Tr}\rho_A^2 = \frac{1}{2}\text{Tr}(\rho \otimes \rho K_1)$, and $1 - \text{Tr}\rho_B^2 = \frac{1}{2}\text{Tr}(\rho \otimes \rho K_2)$, so we can obtain the term $\sqrt{(1 - \text{Tr}\rho_A^2)(1 - \text{Tr}\rho_B^2)}$, provide that we have two copies of the state, see [24, 25].

On the other hand, the term $\|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\|$ can be obtained by generalized entanglement witness [27, 31]. Entanglement witnesses (EW) are Hermitian operators that have positive averages on all separable states, but a negative one on at least one entangled state. It was shown that, a state is entangled if and only if it is detected by some EW [27]. EW can be measured locally, and one can optimize such measurements in various aspects [32]. Nowadays, entanglement witnesses are routinely used in experiments to detect entanglement (see e.g., [33]). Using the similar method of [34], we can directly measure the value of $\|\mathcal{R}(\rho - \rho_A \otimes \rho_B)\|$ by a generalized witness. The method is as follows: from the result of matrix analysis [35], every operator has a singular value decomposition (SVD), so we can get the SVD

of $R(\rho - \rho_A \otimes \rho_B)$ as $R(\rho - \rho_A \otimes \rho_B) = UDV^+$, with U, V unitary matrices, D is a diagonal matrix. Then since the trace norm has a variational representation as $\|A\| = \max\{|\text{Tr}[X^+AY]| \}$, where X, Y are unitary matrices, we can get that $\|R(\rho - \rho_A \otimes \rho_B)\| = \text{Tr}[VU^+R(\rho - \rho_A \otimes \rho_B)]$. Now the only problem leaved is how to deal with the realignment operation. Note that every map ϕ on inner product space can induce its adjoint map ϕ^* , i.e., $\text{Tr}[\phi(X)Y^+] = \text{Tr}[X(\phi^*(Y))^+]$, and for our question, it reads that $\text{Tr}[VU^+R(\rho - \rho_A \otimes \rho_B)] = \text{Tr}[R^*(VU^+)(\rho - \rho_A \otimes \rho_B)]$, and the adjoint map of the realignment operation is defined by $R^*(VU^+) := [\mathcal{R}^{-1}(VU^+)^T]^T$, where T is the transpose, and \mathcal{R}^{-1} is the inverse map of the realignment operation. Now define $W := [\mathcal{R}^{-1}(VU^+)^T]^T$, then we get that $\|R(\rho - \rho_A \otimes \rho_B)\| = \text{Tr}[W(\rho - \rho_A \otimes \rho_B)]$, which is clearly physical accessible (e.g., see [33]).

Next we consider some examples to illustrate further the tightness and significance of our bound. To show that our bound is close to the real concurrence, note that in [38], the authors find a fast optimal algorithm to calculate the entanglement of formation of a mixed state, so we can use the optimal algorithm of [38] to give the estimation of the concurrence, and comparing it with our bound. The concurrence using optimal method of [38] is represented by blue colors, our low bound is represented by green colors.

Example 1 Isotropic states are a class of $U \otimes U^*$ invariant mixed states in $d \times d$ systems

$$\rho_F = \frac{1-F}{d^2-1} (I - |\Psi^+\rangle\langle\Psi^+|) + F|\Psi^+\rangle\langle\Psi^+|, \quad (12)$$

where $|\Psi^+\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ and $F = \langle\Psi^+|\rho_F|\Psi^+\rangle$, satisfying $0 \leq F \leq 1$, is the *fidelity* of ρ_F and $|\Psi^+\rangle$.

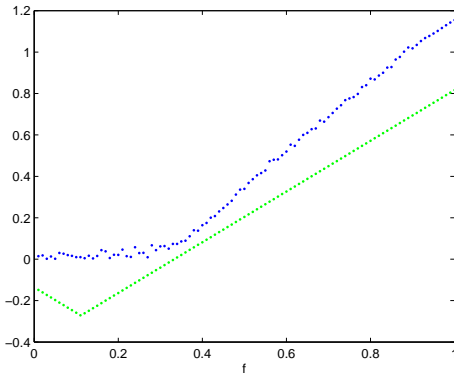


FIG. 1: Figure of example 1

Example 2. Paweł Horodecki introduced a 3×3 bound entangled state in Ref. [28], and the density ma-

trix ρ is real and symmetric,

$$\rho = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}, \quad (13)$$

where $0 < a < 1$. Let us consider a mixture of this state with white noise,

$$\rho(p) = p\rho + (1-p)\frac{\mathbb{1}}{9}, \quad (14)$$

and show the curves $1 - \|R(\rho)\| = 0$, $1 - \|\tau\| - (\text{Tr}\rho_A^2 + \text{Tr}\rho_B^2)/2 = 0$, $\sqrt{(1 - \text{Tr}\rho_A^2)(1 - \text{Tr}\rho_B^2)} - \|R(\rho - \rho_A \otimes \rho_B)\| = 0$ with respect to the CCNR criterion, its optimal nonlinear witness, and Theorem 1 in Fig. 2. It is found that the state $\rho(p)$ still has entanglement when $p = 0.9955$, $a = 0.236$, using the CCNR criterion. According to Theorem 1, one can obtain an upper bound $p = 0.9939$, $a = 0.232$ for $\rho(p)$ which is still entangled.

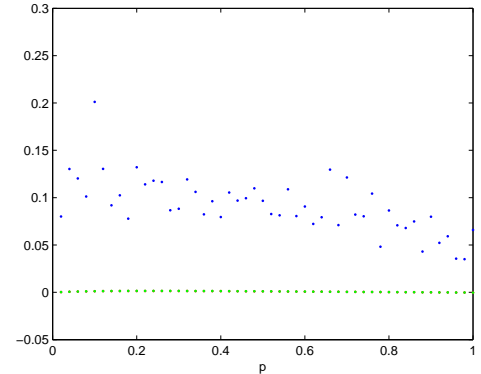


FIG. 2: Figure of example 2

When $p = 0.9955$, $a = 0.236$, using steepest descent method, we get the concurrence is 0.101855, using our theorem, the lower bound is 0.000487. When $p = 0.9939$, $a = 0.232$, using steepest descent method, we get the concurrence is 0.101758, using our theorem, the lower bound is 0.000019.

Example 3. Consider the following states introduced in Ref. [39]:

$$\rho(\alpha) = \frac{2}{7}|\Psi^+\rangle\langle\Psi^+| + \frac{2}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \quad (15)$$

where $2 \leq \alpha \leq 5$, $\sigma_+ := \frac{1}{3}(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|)$, $\sigma_- := \frac{1}{3}(|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|)$.

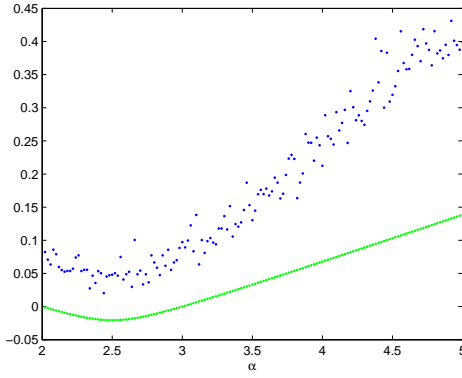


FIG. 3: Figure of example 3

In summary, we have provided an analytical formula for a lower bound of concurrence, by finding a connection with the currently most powerful detection criterion. The bound is very close to the actual values of concurrence for some special class of quantum states. Also, this bound is experimentally implementable and computationally very efficient, allowing to not only detect, but also to quantify entanglement in an experimental scenario.

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